

# Massive Scalar Field Quantum Cosmology

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We study the third quantized formulation of the massive scalar field quantum cosmology for the Friedmann-Robertson-Walker universe. The Hamiltonian is equivalent to an infinite number of coupled oscillators whose couplings and frequencies are intrinsic time-dependent. We propose the invariant operators whose eigenstates provide the exact wave functions of the universe.

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## I. INTRODUCTION

In quantum cosmology both spacetime geometries and matter fields are quantized and obey the Wheeler-DeWitt (WDW) equation. Minisuperspace quantum cosmological models [1, 2] have a few degrees of freedom and hence allow one to find wave functions exactly or approximately, though the solution space of the WDW equation is infinite dimensional. The geometry of the Friedmann-Robertson-Walker (FRW) universe is prescribed by a single scale factor and is thus the simplest minisuperspace cosmological model. The homogeneous and isotropic universe of FRW geometry together with a homogeneous scalar field is still favored by the current observational data. In fact, the inflationary cosmology with a large-field inflaton with power law less than two is consistent with nine-year WMAP data [3] and Planck 2013 results [4].

The massive scalar field quantum cosmology (MSQC), which naturally includes fluctuations of geometry, may be a viable inflationary model. The MSQC has only two degrees of freedom: the scale factor of the FRW geometry and the homogeneous scalar field. However, MSQC is not just a quantum mechanical system since it is governed by a relativistic wave equation in the minisuperspace of the three-geometry and the scalar field. Furthermore, the scalar field is parametrically coupled to the scale factor in the WDW equation, which is analogous to the Klein-Gordon equation with a time-dependent mass.

The parametric interaction of the massive scalar field with the scale factor prevents the WDW equation from being simply separated by the harmonic wave functions for the scalar field. Provided that the scale factor is used as an intrinsic time, the WDW equation has the Cauchy initial value problem and is equal to a time-dependent matrix equation, whose solutions show couplings among harmonic wave functions for the massive scalar field during the evolution [5, 6]. The Feshbach-Villars equation [7] allows one to express the WDW equation in the first order formalism, which has the same form as the Schrödinger equation with a non-Hermitian Hamiltonian [8, 9], which does not, however, possess any quantum invariant that has been found for time-dependent oscillators by Lewis and Riesenfeld [10].

In this paper we show that MSQC in the third quantized formulation is equivalent to an infinite number of coupled oscillators with intrinsic time-dependent frequencies. We further advance quantum invariants for MSQC in the third quantized formulation that play the role of time-dependent annihilation and creation operators in constructing all quantum states, which generalize those for decoupled time-dependent oscillators [11–13].

The organization of this paper is as follows. In Sec. II we review the WDW equation for MSQC in the first order formalism and show that the WDW equation does not have any quantum invariant in the first order formalism. In Sec. III we formulate the WDW equation in the third quantization and express the Hamiltonian in the third quantization as an infinite number of coupled time-dependent oscillators. We advance the invariant operators and discuss their physical implications.

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## II. FIRST ORDER FORMALISM FOR WDW EQUATION

The WDW equation for the FRW universe with a massive scalar field is [in units of  $c = \hbar = G = 1$ ]

$$\left[ \frac{\partial^2}{\partial \alpha^2} + \hat{H}_m(\phi, \alpha) \right] \Psi(\alpha, \phi) = 0, \quad (1)$$

where  $e^\alpha$  is the scale factor and

$$\hat{H}_m(\alpha) = -\frac{\partial^2}{\partial \phi^2} + m^2 e^{6\alpha} \phi^2 - k e^{4\alpha}, \quad (2)$$

is the Hamiltonian for the massive scalar field. Here the last term comes from the scalar curvature of the three-geometry with  $k = 1, 0$  and  $-1$  for a closed, spatially flat and open universe, respectively. The Hamiltonian (2) describes a harmonic oscillator and has an  $su(1, 1)$  algebra [14].

The wave functions for Eq. (1) have not been known in spite of intensive investigations. The mathematical difficulty lies in the parametric dependence of the Hamiltonian (2), in which the frequency depends on the intrinsic time  $\alpha$ . It is worthy to mention that scalar quantum electrodynamics in the vector potential  $\vec{A}(t) = \vec{B}(t) \times \vec{x}/2$  has a similar structure as the WDW equation [15]. In contrast to parametrically interacting relativistic equations, time-dependent harmonic oscillators have a quadratic invariant found by Lewis and Riesenfeld, whose eigenstates provide exact quantum states up to time-dependent phase factors [10].

Hence it is interesting to investigate whether the WDW equation in the first order formalism possesses such a quantum invariant or not. Employing the Feshbach-Villars equation for a scalar field [7], Mostafazadeh expressed Eq. (1) in the first order formalism [8, 9]

$$i \frac{\partial}{\partial \alpha} \begin{pmatrix} \Psi + i \frac{\partial \Psi}{\partial \alpha} \\ \Psi - i \frac{\partial \Psi}{\partial \alpha} \end{pmatrix} = \frac{i}{2} \begin{pmatrix} 1 + \hat{H}_m & -(1 - \hat{H}_m) \\ 1 - \hat{H}_m & -(1 + \hat{H}_m) \end{pmatrix} \begin{pmatrix} \Psi + i \frac{\partial \Psi}{\partial \alpha} \\ \Psi - i \frac{\partial \Psi}{\partial \alpha} \end{pmatrix}. \quad (3)$$

Though Eq. (3) may be interpreted as the Schrödinger equation with the non-Hermitian Hamiltonian

$$\mathbf{H} = \frac{i}{2} \begin{pmatrix} 1 + \hat{H}_m & -(1 - \hat{H}_m) \\ 1 - \hat{H}_m & -(1 + \hat{H}_m) \end{pmatrix}, \quad (4)$$

there does not exist any invariant in the algebra of  $su(2) \otimes su(1, 1)$ , which satisfies

$$i \frac{\partial \mathbf{I}}{\partial \alpha} + [\mathbf{I}, \mathbf{H}] = 0. \quad (5)$$

On the other hand, Kim introduced another form of the first order formalism [5, 6]

$$\frac{\partial}{\partial \alpha} \begin{pmatrix} \Psi \\ \frac{\partial \Psi}{\partial \alpha} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\hat{H}_m & 0 \end{pmatrix} \begin{pmatrix} \Psi \\ \frac{\partial \Psi}{\partial \alpha} \end{pmatrix}. \quad (6)$$

A stratagem is to use the eigenstates of the massive scalar field

$$\hat{H}_m |\Phi_n(\phi; \alpha)\rangle = m e^{3\alpha} (2n + 1) |\Phi_n(\phi; \alpha)\rangle \quad (7)$$

and to expand the wave function in the form

$$|\Psi(\alpha, \phi)\rangle = |\vec{\Phi}(\phi; \alpha)\rangle^T \cdot \vec{\psi}(\alpha), \quad (8)$$

where  $|\vec{\Phi}(\phi; \alpha)\rangle$  is a column vector of eigenstates and  $T$  denotes the transpose. Then, the wave function in the first order formalism can be solved in terms of the scattering matrix [6]

$$\begin{pmatrix} |\Psi(\alpha, \phi)\rangle \\ \frac{\partial}{\partial \alpha} |\Psi(\alpha, \phi)\rangle \end{pmatrix} = \begin{pmatrix} |\vec{\Phi}(\phi; \alpha)\rangle^T & 0 \\ 0 & |\vec{\Phi}(\phi; \alpha)\rangle^T \end{pmatrix} \mathcal{T} \exp \left[ \int_{\alpha_0}^{\alpha} \begin{pmatrix} \Omega & I \\ -E(\alpha) & \Omega \end{pmatrix} d\alpha \right] \begin{pmatrix} \vec{\psi}(\alpha_0) \\ \frac{d\vec{\psi}(\alpha_0)}{d\alpha} \end{pmatrix}. \quad (9)$$

Here the Cauchy initial data are

$$\begin{aligned} \vec{\psi}(\alpha_0) &= \int d\phi \langle \vec{\Phi}(\phi; \alpha_0) | \Psi(\alpha_0, \phi) \rangle, \\ \frac{d\vec{\psi}(\alpha_0)}{d\alpha} &= \int d\phi \langle \vec{\Phi}(\phi; \alpha_0) | \frac{\partial}{\partial \alpha} \Psi(\alpha_0, \phi) \rangle, \end{aligned} \quad (10)$$

and  $\mathcal{T}$  denotes the time-ordered integral and in the number-state representation (7) the coupling matrix and the energy-eigenvalue matrix are given by

$$\Omega = \frac{3}{4}(\hat{a}^2 - \hat{a}^{+2}), \quad E(\alpha) = me^{3\alpha}(2\hat{a}^+ \hat{a} + 1) - ke^{4\alpha}. \quad (11)$$

The off-diagonal matrix  $\Omega$  causes continuous transitions among different number states of the massive scalar field. Furthermore, in the limit of  $\alpha$ ,  $\alpha_0 = -\infty$  with a finite difference  $\Delta\alpha = \alpha - \alpha_0$ ,  $E(\alpha)$  is exponentially suppressed, so the scattering matrix simply becomes

$$\mathcal{T} \exp \left[ \int_{\alpha_0}^{\alpha} \begin{pmatrix} \Omega & I \\ -E(\alpha) & \Omega \end{pmatrix} d\alpha \right] = \exp \left[ \begin{pmatrix} \Omega & I \\ 0 & \Omega \end{pmatrix} (\alpha - \alpha_0) \right] = \begin{pmatrix} e^{\Omega\Delta\alpha} & \Delta\alpha e^{\Omega\Delta\alpha} \\ 0 & e^{\Omega\Delta\alpha} \end{pmatrix}. \quad (12)$$

Therefore, the wave function near the big bang singularity infinitely oscillates as

$$|\Psi(\alpha, \phi)\rangle = |\vec{\Phi}(\alpha)\rangle^T e^{\Omega\Delta\alpha} \left[ \vec{\psi}(\alpha_0) + \Delta\alpha \frac{\partial \vec{\psi}(\alpha_0)}{\partial \alpha} \right]. \quad (13)$$

However, the magnitude of the wave function

$$\langle \Psi(\alpha, \phi) | \Psi(\alpha, \phi) \rangle = \left| \vec{\psi}(\alpha_0) + \Delta\alpha \frac{\partial \vec{\psi}(\alpha_0)}{\partial \alpha} \right|^2 \quad (14)$$

is determined only by the Cauchy data, independently of the evolution of the scalar field and the geometry, which was first observed in Ref. [6]. This implies that any regular wave function should have this scale invariance near the big bang singularity.

### III. THIRD QUANTIZED FORMULATION

The WDW equation (1) for the FRW universe is a hyperbolic equation, in particular, when  $\alpha$  is regarded as a time-like variable in the minisuperspace of the metric and the field. In general, the superspace for a globally hyperbolic spacetime with or without matter fields has a Lorentzian signature. The WDW equation is thus analogous to a Klein-Gordon equation, in which one of the superspace variables plays the role of the intrinsic time. It should be noted, however, that the time cannot be uniquely determined: any functional of the chosen time-like variable also becomes another time-like variable. But the hyperbolic nature of the WDW equation does not depend on the redefinition of the time-like variable. The same conceptual problem occurs in the foliation of a hyperbolic spacetime. In this paper we adopt the picture that  $\alpha$  is an intrinsic time in the WDW equation for MSQC.

In the third quantized formulation [16–20], the WDW equation for MSQC derives from the action in the minisuperspace

$$\mathcal{S} = \frac{1}{2} \int d\alpha d\phi \left[ \left( \frac{\partial \Psi}{\partial \alpha} \right)^2 - \left( \frac{\partial \Psi}{\partial \phi} \right)^2 - V(\phi, \alpha) \Psi^2 \right], \quad (15)$$

where

$$V(\phi, \alpha) = m^2 e^{6\alpha} \phi^2 - ke^{4\alpha}. \quad (16)$$

The variation  $\delta\mathcal{S}/\delta\Psi$  leads to the WDW equation (1). We shall focus on the massive scalar field and consider the massless scalar field as the limiting case of  $m = 0$ .

To find the Hamiltonian from the action (15), we expand the wave function according to Eq. (8) to get

$$\int d\phi \Psi^2 = \vec{\psi}^T \cdot \vec{\psi}, \quad (17)$$

and

$$\int d\phi \left( \frac{\partial \Psi}{\partial \alpha} \right)^2 = (\dot{\vec{\psi}}^T + \vec{\psi}^T \Omega^T) \cdot (\dot{\vec{\psi}} + \Omega \vec{\psi}), \quad (18)$$

where  $\Omega$  is the coupling matrix (11). Hence the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2} (\dot{\vec{\psi}}^T + \vec{\psi}^T \Omega^T) \cdot (\dot{\vec{\psi}} + \Omega \vec{\psi}) - \frac{1}{2} \vec{\psi}^T E(\alpha) \vec{\psi} \quad (19)$$

Introducing the canonical momentum vector

$$\vec{\pi} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{\psi}}^T} = \dot{\vec{\psi}} + \Omega \vec{\psi} \quad (20)$$

and using  $\Omega^T = -\Omega$ , we obtain the Hamiltonian

$$\mathcal{H}(\alpha) = \frac{1}{2} \vec{\pi}^T \cdot \vec{\pi} - \vec{\pi}^T \Omega \vec{\psi} + \frac{1}{2} \vec{\psi}^T E \vec{\psi}. \quad (21)$$

The quantum law for the universe is the functional Schrödinger equation

$$i \frac{\partial}{\partial \alpha} \Psi(\alpha) = \hat{\mathcal{H}}(\alpha) \Psi(\alpha). \quad (22)$$

The Hamiltonian (21) is an infinite system of coupled oscillators with time-dependent frequencies  $E(\alpha)$ . We propose a pair of invariant operators of the form

$$\begin{aligned} \vec{\mathcal{A}}(\alpha) &= i[\mathcal{O}\vec{\pi} - (\dot{\mathcal{O}} + \mathcal{O}\Omega^T)\vec{\psi}], \\ \vec{\mathcal{A}}^\dagger(\alpha) &= -i[\vec{\pi}^\dagger \mathcal{O}^\dagger - \vec{\psi}^\dagger (\dot{\mathcal{O}}^\dagger + \Omega^* \mathcal{O}^\dagger)], \end{aligned} \quad (23)$$

which satisfy the Liouville-von Neumann equation

$$i \frac{\partial}{\partial \alpha} \begin{pmatrix} \vec{\mathcal{A}}(\alpha) \\ \vec{\mathcal{A}}^\dagger(\alpha) \end{pmatrix} + \left[ \begin{pmatrix} \vec{\mathcal{A}}(\alpha) \\ \vec{\mathcal{A}}^\dagger(\alpha) \end{pmatrix}, \hat{\mathcal{H}}(\alpha) \right] = 0. \quad (24)$$

Here  $\mathcal{O}$  is a matrix satisfying

$$\ddot{\mathcal{O}} + 2\dot{\mathcal{O}}\Omega^T + \mathcal{O}(E + \dot{\Omega}^T + (\Omega^T)^2) = 0, \quad (25)$$

and is chosen such that  $\vec{\mathcal{A}}$  and  $\vec{\mathcal{A}}^\dagger$  satisfy the equal-time commutator

$$[\vec{\mathcal{A}}(\alpha), \vec{\mathcal{A}}^\dagger(\alpha)] = I. \quad (26)$$

It is interesting to compare Eq. (25) with the WDW equation in the vector notation

$$\ddot{\vec{\psi}} - 2\Omega \dot{\vec{\psi}} + (E - \dot{\Omega} + (\Omega)^2)\vec{\psi} = 0. \quad (27)$$

In the special case of a massless scalar field ( $m = 0$ ), we take the limit  $\Omega = 0$ , which has been elaborated in Ref. [21]. The Hamiltonian, after decomposing by the Fourier modes  $|\Phi\rangle = e^{ip\phi}$ , is another infinite system of decoupled time-dependent oscillators, so the wave function is the product of wave functions for Fourier cosine and sine modes:

$$\Psi(\alpha, \phi) = \prod_{(\pm)p} \Psi_{(\pm)p}(\alpha) e^{ip\phi}, \quad (28)$$

where

$$i \frac{\partial}{\partial \alpha} \Psi_{(\pm)p}(\alpha) = \hat{\mathcal{H}}_{(\pm)p}(\alpha) \Psi_{(\pm)p}(\alpha). \quad (29)$$

The invariant operators  $\vec{\mathcal{A}}$  and  $\vec{\mathcal{A}}^\dagger$  decouple among themselves and are given by

$$\begin{aligned} \hat{\mathcal{A}}_{(\pm)p}(\alpha) &= i[u_{(\pm)p}^* \hat{\pi}_{(\pm)p} - \dot{u}_{(\pm)p}^* \hat{\psi}_{(\pm)p}], \\ \hat{\mathcal{A}}_{(\pm)p}^\dagger(\alpha) &= -i[u_{(\pm)p} \hat{\pi}_{(\pm)p} - \dot{u}_{(\pm)p} \hat{\psi}_{(\pm)p}], \end{aligned} \quad (30)$$

where  $u_{(\pm)p}$  satisfies the mode equation

$$\ddot{u}_{(\pm)p} + (p^2 - ke^{4\alpha})u_{(\pm)p} = 0, \quad (31)$$

and the Wronskian condition  $\text{Wr}[u_{(\pm)p}, u_{(\pm)p}^*] = i$ . The physical implications of the third quantized formulation have been discussed in detail in Ref. [21].

## IV. CONCLUSION

In this paper we have investigated the massive scalar field quantum cosmology in the view of nine-year WMAP data [3] and Planck 2013 results [4]. Though the massive scalar field alone seems to be likely excluded, the curvature-square model by Starobinsky turns out a viable scenario [3, 4]. As the curvature-square term originates from spacetime fluctuations probed by matter fields or gravity itself, it would be worthy to study quantum cosmology with a massive scalar field. In fact, the Wheeler-DeWitt equation necessarily involves quantum fluctuations of the geometries and the wave functions contain higher curvature effects. The de Broglie-Bohm interpretation of the wave function results in semiclassical quantum gravity with non-trivial higher curvature terms when the Planckian scale gravity separates from the massive scalar field via the Born-Oppenheimer idea [22, 23].

The wave function for the massive scalar field quantum cosmology could be separated as a vector equation due to the continuous transitions among the eigenstates of the massive scalar field, which depends parametrically on the intrinsic time [5, 6]. This coupling matrix survives as an effective gauge potential in semiclassical quantum gravity [22, 23]. In the first order formalism the Wheeler-DeWitt equation could be analytically expressed by introducing the scattering matrix, which provides a perturbation method for wave functions. However, in the third quantized formulation the massive scalar field quantum cosmology could be expressed as an infinite system of coupled intrinsic time-dependent oscillators as shown in this paper. We have advanced the invariant operators for the resulted coupled oscillators, which will provide the exact wave functions of the universe. The quantum nature of the universe will be further studied from the view of the recent precision observation of CMB data in a future publication.

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